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Quantum propagator for a nonrelativistic particle in the vicinity of a time machine

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We study the propagator of a nonrelativistic, noninteracting particle in any nonrelativistic "time-machine" spacetime of the following type: an external, chronal spacetime in which two spatial regions V_- at time t_- and V_+ at time t_+ are connected by two temporal wormholes, one leading from the past side of V_- to the future side of V_+ and the other from the past side of V_+ to the future side of V_-. We express the propagator explicitly in terms of those for the chronal spacetime and for the two wormholes; and from that expression we show that the propagator satisfies completeness and unitarity in the initial and final "chronal regions" (regions without closed timelike curves) and its propagation from the initial region to the final region is unitary. However, within the time machine it satisfies neither completeness nor unitarity. We also give an alternative proof of initial-region-to-final-region unitarity based on a conserved current and Gauss's theorem. This proof can be carried over without change to most any nonrelativistic time-machine spacetime and it is valid as long as the particle is not interacting with itself or any other quantum particle; it can, however, interact with an external field (garvitational or otherwise). This result is the nonrelativistic version of a theorem by Friedman, Papastamatiou, and Simon, which says that for a free scalar field quantum-mechanical unitarity follows from the fact that the classical evolution preserves the Klein-Gordon inner product.

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I. INTRODUCTION

Spacetime and the phenomenon of gravitation are described very well at a classical level by the theory of general relativity. Locally, spacetime is isomorphic to Minkowski space and there is a well-defined light cone and microscopic causality. Globally, however, things may be quite different. There is nothing in the laws of classical general relativity that prevents spacetimes from having closed causal (timelike or null) curves, that is, future-directed curves through a point p such that if one travels along them always toward the local future one returns to the same spacetime point. It is easy to find examples of spacetimes in which closed timelike curves have always existed [1]. None of these examples, generally referred to as eternal time machines, look very much like our Universe. In each of these cases, it is not possible to pose the Cauchy problem in the usual manner, at an arbitrary "initial moment of time," for matter fields propagating in these spacetimes [2], and one can therefore expect that these spacetimes are rather pathological.

Another type of causality violation is one in which closed timelike curves develop during the evolution of a spacetime from some reasonable initial conditions. An example of such behavior is found in the Kerr solution which is believed to be the end point of gravitational collapse with rotation. The region in which causality violation occurs is close to the singularity and interior to the inner horizon. It might be the case that such behavior is generic under certain circumstances, as Tipler [3] has shown that if matter obeys the weak energy condition, and closed timelike curves develop to the future of some Cauchy surface, then the spacetime must be geodesically incomplete. We conjecture that if the weak energy condition is satisfied in an asymptotically flat spacetime, the closed timelike curves will also only occur in the interiors of horizons and the physics exterior to any horizon will always be unaffected by the paradoxes and difficulties associated with closed timelike curves. Hawking [4] has proposed the chronology protection conjecture, presently still unproven, which would prevent closed timelike curves under a wide range of circumstances.

Systems that obey the weak energy condition classically, for example, a free scalar field, do not necessarily obey it after quantization [5,6]. Under these circumstances, it may be possible to create a region of spacetime that includes closed timelike curves (a *dischronal region*) without the occurrence of spacetime singularities other

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than those associated with the chronology horizon. Similarly, since the laws of physics are time reversal invariant, we expect that dischronal regions could disappear. Any spacetime in which a dischronal region is preceded and followed by *chronal regions* (regions without closed timelike curves) will be referred to in this paper as a "time machine." Morris, Thorne, and Yurtsever [7] have shown that one way such spacetimes can arise is from the relative motion of the mouths of a spatial wormhole that initially connects two spacelike separated regions in Minkowski space.

There are a number of undesirable, apparent paradoxes that arise in such a spacetime. Recently several authors [8,9] have discussed the resolutions of some of these paradoxes within the realm of classical physics. Despite these resolutions, the Cauchy problem fails to be well posed for classical interacting systems (e.g., "billiard balls") in the presence of time machines [8]: For the classical initial value problem, there can exist an infinite number of consistent (i.e., nonparadoxial) classical evolutions. In other words, although the paradoxes can be avoided, predictability is still violated, classically.

Not so quantum mechanically. Quantum mechanics restores predictability in the usual, probabilistic sense [10–12]: each of the allowed classical evolutions acquires, in the WKB approximation, a finite, predictable probability of being followed. However, predictability is restored only at a price: in the presence of a time machine, quantum mechanics does *not* retain all of the "nice" features that we normally are accustomed to. For example, the propagator which takes an evolving system from initial conditions before the dischronal region to a final state afterward is *not,* in general, unitary [11–14]. On the other hand, when the system being evolved is noninteracting (free), the evolution *is* unitary, at least for those examples that have been studied: Friedman, Papastamatiou, and Simon [11] have shown that a relativistic, free scalar field evolves unitarily in *any* time machine spacetime that is initially static and finally static; and Politzer [14] has proved unitarity for a nonrelativistic, free particle in a particular time machine: flat spacetime with identification of identical, finite sized regions at different times.

The generality of unitarity for a free scalar field suggests that, similarly, the nonrelativistic free particle should evolve unitarily in most any time-machine spacetime. That it does, indeed, do so we shall demonstrate in this paper. In fact our proof is slightly more general: it is valid for any noninteracting particle, i.e., a particle that does not interact with other quantum particles, but that can interact with external fields (gravitational or otherwise).

Among all standard formulations of quantum mechanics, the conceptually simplest and most familiar is the Hamiltonian formulation in the Schrödinger picture; there the state at time $t, |\psi(t)\rangle$, is determined by evolving an initial state $|\psi(0)\rangle$ with the Hamiltonian operator $H(t)$. We cannot use this Hamiltonian formulation in the presence of a time machine (or more generally in any nonglobally hyperbolic spacetime), because such a spacetime cannot be globally foliated by spacelike surfaces of constant "time" t, and correspondingly the standard notion of "state at time t," $|\psi(t)\rangle$, does not exist; see, e.g., [12] and references therein.

The only standard formulation of quantum mechanics that seems to survive in a nonglobally hyperbolic spacetime is Feynman's path-integral formulation [11,12]. The path-integral formulation can be derived from the Hamiltonian formulation and conversely, for certain classes of Hamiltonian, provided that the Hamiltonian formulation exists [15]. However, in the absence of a Hamiltonian formulation, the path integral is the only foundational tool available for quantum theory.

In this paper we shall study a noninteracting, nonrelativistic particle. We being by recalling a few key features of such a particle's quantum mechanical description in a globally hyperbolic spacetime (no time machine). Suppose that at a time t_i, the particle is in an eigenstate of position at x_i, which we denote $|i,t_i\rangle$. The propagator then is the amplitude $\mathcal{G}_{ji}$ given by

$$\mathcal{G}_{ji}=\langle j,t_j|i,t_i\rangle=\sum\exp[iS_{ji}/\hbar]\ , \tag{1}$$

where the summation is over all paths from (x_i,t_i) to (x_j,t_j) and S_{ji} is the classical action evaluated along the path in question.

In a globally hyperbolic spacetime, the propagator obeys the group properties of completeness,

$$\mathcal{G}_{ji}=\sum_k \mathcal{G}_{jk}\mathcal{G}_{ki}\ ,\quad t_j\geq t_k\geq t_i\ , \tag{2}$$

and unitarity [16],

$$\sum_k \mathcal{G}^*_{ki}\mathcal{G}_{kj}=\begin{cases}\delta_{ij} & \text{if } t_i=t_j<t_k\ ,\\ \mathcal{G}^*_{ji} & \text{if } t_i<t_j<t_k\ ,\\ \mathcal{G}_{ij} & \text{if } t_j<t_i<t_k\ .\end{cases} \tag{3}$$

Completeness asserts that if one examines $\mathcal{G}_{ik}$, then the particle will have been at some position at any intermediate time t_j. Unitarity is the statement that it is possible to reverse the time evolution of a system so as to reconstruct an earlier state of the system given the state at a later instant of time. In a globally hyperbolic spacetime, unitarity can be viewed as equivalent to conservation of probability. It should be noted that completeness and unitarity ensure that the time evolution of a system is described by elements of a group, since the additional axiom of associativity is clearly satisfied as a consequence of (2). In the Hamiltonian formulation, completeness and unitarity are trivially guaranteed by Hermiticity of the Hamiltonian, as H is the generator of the Lie algebra associated with the group of time evolution.

As an explicit example consider the propagator K_{ji} of a free nonrelativistic particle of mass m propagating in the standard flat spacetime of nonrelativistic physics [15]:

$$K_{ji}=\begin{cases}\left[\dfrac{m}{2\pi i\hbar(t_j-t_i)}\right]^{3/2}\exp\left[\dfrac{im(x_j-x_i)^2}{2\hbar(t_j-t_i)}\right] & \text{if } t_j>t_i\ ,\\ 0 \quad \text{if } t_j<t_i\ .\end{cases} \tag{4}$$

K_{ji} vanishes if $t_j < t_i$ since the nonrelativistic particle propagates only to the future. Since we are dealing with a nonrelativistic propagator, the light cone is the line $t = \text{const}$; i.e., a particle located at (x_i, t_i) can propagate to any point in which $t > t_i$. Clearly K_{ji} obeys (2) and (3), and can be derived by either Hamiltonian methods or by path integrals.

In a time-machine spacetime, if the evolution from time t_i in the initial chronal region, through the dischronal region, to time t_j in the final chronal region is unitary, then the standard notion of quantum mechanical state and the Hamiltonian formulation of quantum mechanics exist in both chronal regions but not in the dischronal region [11,12], and the unitary propagator $\mathcal{G}_{ji}$ relates the initial and final states in the usual way [Eq. (1)]. If the propagator is not unitary, then its use and the associated formulation of quantum mechanics might be slightly different from what one is accustomed to in globally hyperbolic situations [11,12], but this will not concern us here.

Politzer [14] has shown, for a particular time machine, that the nonrelativistic, free-particle propagator is unitary. In this paper we generalize his result to the broader class of time-machine spacetimes depicted in Fig. 1. Each such spacetime consists of a nonrelativistic, chronal exterior region plus two temporal wormholes. The wormholes connect two different, arbitrarily shaped spatial regions in the exterior spacetime: V_- at time t_- and V_+ at time $t_+ > t_-$. The upper wormhole in the figure connects the bottom (past) face of V_+ to the top (future) face of V_-; i.e., it is a wormhole in which, by traveling forward in local time, one travels backward in external time from t_+ to t_-. The lower wormhole connects the bottom (past) face of V_- to the top (future) face of V_+, so that by traveling forward in local time through it, one jumps forward in external time from t_- to t_+. The shapes of the wormholes are arbitrary and the space inside them can be curved and time evolving. They might have perfectly reflecting walls, or points on their "walls" might be identified in such a way that the wormholes are spatially closed with no real walls at all. Whatever may be their form, because the spacetime inside them is assumed to be nonsingular and foliable by a family of spacelike hypersurfaces, the path integral will produce a unitary propagator that takes the particle forward in local time from the beginning of each wormhole to its end.

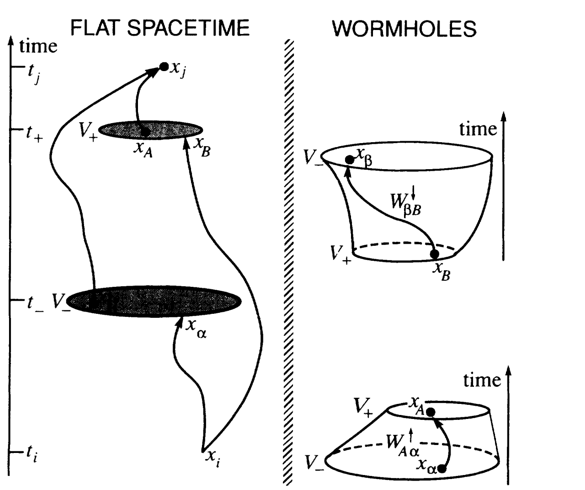

FIG. 1. A nonrelativistic time-machine spacetime and two paths in the spacetime that contribute to the propagator $\mathcal{G}_{ji}$ for a noninteracting, nonrelativistic particle. Left: The external, nonrelativistic, flat part of the spacetime. Right: The temporal wormholes which connect the regions V_- and V_+.

We introduce the notation that Greek indices $\alpha, \beta, \ldots$, denote points in V_- (at t_-) and capital Latin indices $A, B, \ldots$, denote points in V_+ (at t_+), and we denote the unitary propagator from (x_B, t_+) through the upper wormhole to (x_β, t_-) by $W^\downarrow_{\beta B}$, and the unitary propagator from (x_α, t_-) through the lower wormhole to (x_A, t_+) by $W^\uparrow_{A\alpha}$. The arrows on these propagators indicate the direction of propagation relative to exterior time. Politzer's time machine is obtained by choosing V_- and V_+ to be identical in size and shape, and choosing the wormholes to be vanishingly short so the future side of V_- is identified with the past side of V_+ and conversely. In Politzer's time machine the wormhole propagators degenerate to the identity function, $W^\downarrow_{\alpha A} = W^\uparrow_{A\alpha} = \delta_{A\alpha}$.

In Sec. II of this paper we construct the propagator $\mathcal{G}_{ji}$ for any wormhole of the type shown in Fig. 1, expressing it in terms of the unitary propagator K_{ji} of the exterior chronal region and the interior propagators $W^\downarrow_{\alpha A}$ and $W^\uparrow_{A\alpha}$ and we use this expression for $\mathcal{G}_{ji}$ to prove that it is complete and unitary in the initial and final chronal regions, and propagates unitarily from the initial to the final region. K_{ji} is the flat-spacetime propagator if the exterior region is flat. Our results remain valid, however, in any general, chronal, curved spacetime with an external nongravitational potential as long as K_{ji} is unitary, both prior and subsequent to the dischronal region, as would arise under almost any circumstance.

In Sec. III we show that, for the spacetime of Fig. 1, $\mathcal{G}_{ji}$ obeys the Schrödinger equation everywhere (in the dischronal region and the wormholes as well as the chronal regions), and we use that fact plus Gauss's theorem to prove unitarity. This second demonstration of unitarity has the virtue that it generalizes without change to most any time machine. In Sec. IV we present concluding remarks.

II. EXPLICIT EXPRESSION FOR THE PROPAGATOR, AND ITS PROPERTIES

Had there been no wormholes, then the propagator $\mathcal{G}_{ji}$ in the time-machine spacetime would simply become the propagator K_{ji} of the exterior chronal spacetime. However, it is not too hard to evaluate the path integral (1) explicitly in the time-machine spacetime which we are considering, because it is simple to find all possible paths by which a particle can propagate from (x_i, t_i) to (x_j, t_j). We will explicitly calculate $\mathcal{G}_{ji}$ for the case that $t_i < t_- < t_+ < t_j$. The possible paths are then labeled by the number of times n that the particle traverses a wormhole, and the contribution to $\mathcal{G}_{ji}$ from all paths with fixed n is $\mathcal{G}^{(n)}_{ji}$. For $n = 0$ we have

$$\begin{aligned}\mathcal{G}^{(0)}_{ji}&=K_{ji}-\int_{V_+} d^3x_A K_{jA}K_{Ai}-\int_{V_-} d^3x_\alpha K_{j\alpha}K_{\alpha i}\\&\quad+\int_{V_-} d^3x_\alpha\int_{V_+} d^3x_A K_{jA}K_{A\alpha}K_{\alpha i}\\&\equiv K_{ji}-K_{jA}K_{Ai}-K_{j\alpha}K_{\alpha i}+K_{jA}K_{A\alpha}K_{\alpha i}\ ,\end{aligned}\tag{5}$$

where K_{ji} is the ordinary propagator in the exterior spacetime and where in the second equality we adopt the notation, such as the summation convention, that repeated *adjacent* indices are integrated over (thus the indices behave like matrix indices). The first term comes from all paths that go from (x_i,t_i) to (x_j,t_j). The second and third terms, which represent the contribution of all paths from (x_i,t_i) to (x_j,t_j) via V_+, and all paths from (x_i,t_i) to (x_j,t_j) via V_-, respectively, must be subtracted off, since any particle that arrives at (x_A,t_+) will then enter the wormhole instead of traveling directly on to (x_j,t_j), and similarly for particles arriving at (x_α,t_-). In these subtractions we have double counted the paths that in ordinary space would have gone from (x_i,t_i) via V_- to V_+ and then to (x_j,t_j), so those are added in the last term.

For $n=1$ the calculation can be done in much the same way:

$$\begin{aligned}\mathcal{G}^{(1)}_{ji}&=K_{j\alpha}W^{\downarrow}_{\alpha A}K_{Ai}-K_{jA}K_{A\alpha}W^{\downarrow}_{\alpha B}K_{Bi}-K_{j\alpha}W^{\downarrow}_{\alpha A}K_{A\beta}K_{\beta i}\\&\quad+K_{jA}K_{A\beta}W^{\downarrow}_{\beta B}K_{B\alpha}K_{\alpha i}+K_{jA}W^{\uparrow}_{A\alpha}K_{\alpha i}\ .\end{aligned}\tag{6}$$

The first term in (6) is the contribution from a particle traveling once through the upper wormhole. The second, third, and fourth are the same types of corrections as we met in Eq. (5). The last term is the contribution of paths that reach (x_α,t_-), then travel through the lower wormhole to (x_A,t_+) and from there to (x_j,t_j).

Similarly one can construct the general $\mathcal{G}^{(n)}_{ji}$ for paths with n wormhole traversals. Unlike the previous two cases, for $n\ge 2$ we have contributions only from paths that begin by reaching (x_A,t_+):

$$\begin{aligned}\mathcal{G}^{(n)}_{ji}&=K_{j\alpha}\underbrace{W^{\downarrow}_{\alpha A}K_{A\beta}W^{\downarrow}_{\beta B}K_{B\gamma}W^{\downarrow}_{\gamma C}\cdots K_{C\delta}}_{(n-1)\text{ times}}W^{\downarrow}_{\delta D}K_{Di}\\&\quad-K_{jA}\underbrace{K_{A\alpha}W^{\downarrow}_{\alpha B}K_{B\beta}W^{\downarrow}_{\beta C}\cdots K_{C\gamma}W^{\downarrow}_{\gamma D}}_{n\text{ times}}K_{Di}\\&\quad+K_{jA}\underbrace{K_{A\alpha}W^{\downarrow}_{\alpha B}K_{B\beta}W^{\downarrow}_{\beta C}\cdots K_{C\gamma}W^{\downarrow}_{\gamma D}}_{n\text{ times}}K_{D\delta}K_{\delta i}\\&\quad-K_{j\alpha}\underbrace{W^{\downarrow}_{\alpha B}K_{B\beta}W^{\downarrow}_{\beta C}K_{C\gamma}\cdots W^{\downarrow}_{\gamma D}K_{D\delta}}_{n\text{ times}}K_{\delta i}\ .\end{aligned}\tag{7}$$

The complete propagator can now be evaluated in terms of K_{ji} by summing over all $\mathcal{G}^{(n)}$:

$$\begin{aligned}\mathcal{G}_{ji}=\sum_{n=0}^{\infty}\mathcal{G}^{(n)}_{ji}&=K_{ji}-K_{j\alpha}\left\{\delta_{\alpha\beta}+W^{\downarrow}_{\alpha A}\left[\sum_{n=0}^{\infty}(KW^{\downarrow})^n\right]_{AB}K_{A\beta}\right\}K_{\beta i}-K_{jA}\left[\sum_{n=0}^{\infty}(KW^{\downarrow})^n\right]_{AB}K_{Bi}\\&\quad+K_{jA}\left\{W^{\uparrow}_{A\alpha}+\left[\sum_{n=0}^{\infty}(KW^{\downarrow})^n\right]_{BC}K_{C\alpha}\right\}K_{\alpha i}+K_{j\alpha}W^{\downarrow}_{\alpha A}\left[\sum_{n=0}^{\infty}(KW^{\downarrow})^n\right]_{AB}K_{Bi}\ .\end{aligned}\tag{8}$$

The sum $[\sum_{n=0}^{\infty}(KW^{\downarrow})^n]_{AB}$ can be expressed formally as

$$\left[\sum_{n=0}^{\infty}(KW^{\downarrow})^n\right]_{AB}=\left[\frac{1}{\delta-KW^{\downarrow}}\right]_{AB}\ .\tag{9}$$

Collecting terms and using this definition we obtain

$$\begin{aligned}\mathcal{G}_{ji}&=K_{ji}+(K_{j\alpha}W^{\downarrow}_{\alpha A}-K_{jA})\left[\frac{1}{\delta-KW^{\downarrow}}\right]_{AB}\\&\quad\times(K_{Bi}-K_{B\beta}K_{\beta i})+(K_{jA}W^{\uparrow}_{A\alpha}-K_{j\alpha})K_{\alpha i}\ .\end{aligned}\tag{10}$$

The propagator $\mathcal{G}_{ji}$ was derived assuming that $t_i<t_-$ and that $t_j>t_+$. However, (10) holds for all values of t_i and t_j, as can readily be seen by considering all possible time orderings of t_i and t_j relative to t_- and t_+ and by noting that $K_{ba}=0$ if $t_b<t_a$. In other words, *the path-integral-defined propagator* $\mathcal{G}_{ji}=\sum\exp[iS_{ji}/\hbar]$ *can be expressed in the form* (10) *for all pairs of points i,j that reside outside the wormholes.*

Using the same method we can express the time reversed operator $\mathcal{G}^{(R)}_{ij}$ which is the propagator to go backward in time from point j to i, in terms of $K^{(R)}_{ij}$, $W^{\downarrow(R)}_{A\alpha}$, and $W^{\uparrow(R)}_{\alpha A}$. By definition, $\mathcal{G}^{(R)}_{ij}$ is the sum of $\exp(iS_{ij}/\hbar)$ over all paths that start at point j and travel backward in time to point i, and similarly for $K^{(R)}_{ij}$, $W^{\downarrow(R)}_{A\alpha}$, and $W^{\uparrow(R)}_{\alpha A}$. The standard argument, valid whenever the relevant region of spacetime is foliable by spacelike hypersurfaces (so no issues of closed timelike curves arise), shows that $K^{(R)}_{ij}=K^{*}_{ji}$ (i.e, time-reversed propagation is produced by the Hermitian conjugate of K_{ji}), and similarly $W^{\downarrow(R)}_{A\alpha}=W^{\downarrow *}_{\alpha A}$ and $W^{\uparrow(R)}_{\alpha A}=W^{\uparrow *}_{A\alpha}$. These relations for the propagators are referred to as "Hermiticity." Our computation reveals that $\mathcal{G}^{(R)}_{ij}$ is given in terms of $K^{(R)}_{ij}$, $W^{\downarrow(R)}_{A\alpha}$, and $W^{\downarrow(R)}_{\alpha A}$ by an expression identical in form to Eq. (8); and this, together with Hermiticity of K_{ji}, $W^{\downarrow}_{\alpha A}$, and $W^{\uparrow}_{A\alpha}$, implies that $\mathcal{G}^{(R)}_{ij}=\mathcal{G}^{*}_{ji}$. Thus, $\mathcal{G}$ is Hermitian.

To examine the completeness properties of $\mathcal{G}_{ji}$ we need to evaluate $\sum_k\mathcal{G}_{jk}\mathcal{G}_{ki}$ for the various cases of t_j, t_k, and t_i greater than or less than t_- and t_+ and subject to $t_i<t_k<t_j$. We use the completeness properties (2) of K_{ji}, together with $\sum_k K_{Ak}K_{kA}=0$. The latter follows from the fact that either K_{Ak} or K_{kA} vanishes depending on whether t_k is greater than or less than t_+. If $t_k>t_+$ or $t_k<t_-$, completeness of $\mathcal{G}_{ji}$ follows directly from the completeness of K_{ji}. Hence, $\mathcal{G}_{ji}$ obeys the completeness condition if the intermediate surface $(t=t_k)$ is chosen to be either to the past or the future of the time machine. If however $t_-<t_k<t_+$ then

$$\sum_k \mathcal{G}_{jk}\mathcal{G}_{ki} \neq \mathcal{G}_{ji} \tag{11}$$

and completeness fails to be satisfied. This is because the particle can cross such an intermediate surface exterior to the wormholes any number of times, including zero. This violation of completeness seems harmless as it happens only while the time machine is operating and completeness is recovered after the time machine has ceased to exist.

Because $\mathcal{G}_{ji} = K_{ji}$ for (x_i, t_i) and (x_j, t_j) both in the initial chronal region or both in the final chronal region, $\mathcal{G}_{ji}$ is unitary in each of these regions. To check unitarity for (x_i, t_i) in the initial chronal region and (x_j, t_j) in the final chronal region, i.e., for propagation through the time machine, we substitute the full expression for $\mathcal{G}^*_{kj}$ and $\mathcal{G}_{ki}$, given by (10) into the unitarity sum $\sum_k \mathcal{G}^*_{kj}\mathcal{G}_{ki}$ to obtain

$$\sum_k \mathcal{G}^*_{kj}\mathcal{G}_{ki} = \left[K^*_{kj} + (K^*_{k\alpha}W^{*\downarrow}_{\alpha A} - K^*_{kA})\left[\frac{1}{\delta - K^*W^{*\downarrow}}\right]_{AB}(K^*_{Bj} - K^*_{B\beta}K^*_{\beta j}) + (K^*_{kA}W^{*\uparrow}_{A\alpha} - K^*_{k\alpha})K^*_{\alpha j}\right]$$
$$\times \left[K_{ki} + (K_{k\gamma}W^{\downarrow}_{\gamma C} - K_{kC})\left[\frac{1}{\delta - KW^{\downarrow}}\right]_{CD}(K_{Di} - K_{D\delta}K_{\delta i}) + (K_{kC}W^{\uparrow}_{C\gamma} - K_{k\gamma})K_{\gamma i}\right] . \tag{12}$$

The summation over k appears between terms such as $K^*_{kj}K_{k\alpha}$, for which we use the fact that K is the background space propagator and is therefore itself unitary, to obtain $\sum_k K^*_{kj}K_{k\alpha} = K_{j\alpha}$. Using this, the unitarity of $W^{\uparrow}_{A\alpha}$ and $W^{\downarrow}_{\alpha A}$, and the identity

$$(\delta_{AB} - K_{A\alpha}W^{\downarrow}_{\alpha B})\left[\frac{1}{\delta - KW^{\downarrow}}\right]_{BC} = \delta_{AC} , \tag{13}$$

we discover that the unitarity condition is satisfied for arbitrary $W^{\downarrow}$ and $W^{\uparrow}$.

Although $\mathcal{G}_{ji}$ propagates unitarily from the initial chronal region to the final chronal region, its propagation within the time machine (i.e., within the dischronal region) is not unitary, as one can check most easily by choosing $t_- < t_i = t_j < t_k < t_+$.

III. UNITARITY FOR A BROAD CLASS OF TIME MACHINES

We now present an alternative proof of unitarity for propagation from the initial chronal region to the final chronal region. This proof (which is a generalization of one given in the unpublished Ref. [10]) has the virtue that it is valid for any time machine whose spacetime everywhere is locally (not globally) foliable by spacelike hypersurfaces with proper time separations that are independent of spatial location. (This is the typical form of nonrelativistic spacetimes.) If the local spatial coordinates are carried perpendicularly from one hypersurface to the next, then the spacetime metric takes the form

$$ds^2 = -c^2 d\tau^2 + g_{pq}(x,\tau)dx^p dx^q . \tag{14}$$

Here c is the speed of light (which is regarded as arbitrarily large since we are in the nonrelativistic limit). The time-machine spacetime of Fig. 1 has this metric, with $g_{pq} = \delta_{pq}$ and $\tau = t$ in the flat exterior, but not inside the wormholes. Note that, because τ everywhere increases toward the local future, it is not possible to cover the dischronal region by a single coordinate patch of this sort; several are required. We assume (for conceptual simplicity) that, as in Fig. 1, so also for our more general time machine, in the distant-past portion of the initial chronal region, space is flat; and similarly for the distant future of the final chronal region.

In ordinary flat space, the path-integral-defined propagator K_{ai} satisfies the Schrödinger equation in its final point a:

$$-\frac{\hbar}{i}\frac{\partial K_{ai}}{\partial t_a} = -\frac{\hbar^2}{2m}\nabla^2_a K_{ai} . \tag{15}$$

The same infinitesimal-propagation argument that is used to derive this equation in flat space (Sec. 4-1 of Feynman and Hibbs [15]) can be used in spacetimes with the metric (14) to derive the corresponding Schrödinger equation

$$-\frac{\hbar}{i}\frac{1}{g^{1/4}}\frac{\partial g^{1/4}\mathcal{G}_{ai}}{\partial \tau_a} = -\frac{\hbar^2}{2m}\nabla^2_a \mathcal{G}_{ai} . \tag{16}$$

Here $g \equiv \det\|g_{pq}\|$ is the determinant whose square root governs spatial volume elements, ∇^2_a is the covariant, spatial Laplacian at the final point a, and there is no sum over the point a. It should be emphasized that this equation was derived by a local analysis, and it is not necessarily the case that the right-hand side of (16) can be identified with a well-defined global Hamiltonian for the system. The Schrödinger equation (16) is valid inside the time machine as well as outside; closed timelike curves do not affect its path-integral-based derivation. To see that this is indeed the case, recall that Eq. (16) is derived by just varying the final end point of the path integral. Therefore all that is required is that the spacetime be regular in an infinitesimal neighborhood of the point in question.

If the particle was not free, its self-interactions in the dischronal region (e.g., billiard-ball collisions [8]) would produce contributions to the action that invalidate the derivation of the Schrödinger equation [10], and presumably thereby would invalidate the following proof of unitarity.

From the Schrödinger equation (16) one can easily derive the following differential conservation law, which is intimately related to the conservation of the probability current:

$$\nabla_a \left[\frac{i\hbar}{2m} (\mathcal{G}_{ai} \nabla_a \mathcal{G}^*_{aj} - \mathcal{G}^*_{aj} \nabla_a \mathcal{G}_{ai}) \right] + \frac{1}{g^{1/2}} \frac{\partial (g^{1/2} \mathcal{G}^*_{aj} \mathcal{G}_{ai})}{\partial \tau_a} = 0 \; . \quad (17)$$

Here ∇_a is the covariant spatial gradient in the metric g_{pq} and there is no sum over the point a. We choose the points i and j to lie in the initial, flat-space chronal region at times t_i and t_j; and we let t_m be a time to the future of t_i and t_j but still in the initial, flat, chronal region, and t_k be a time in the final, flat chronal region. We then construct the volume integral of the conservation law (17) over the spacetime region between t_m and t_k, apply Gauss's theorem to convert the spacetime volume integral into a surface integral, and thereby obtain an integral conservation law. (Recall that Gauss's theorem is valid independently of the topology of the spacetime; it only requires orientability and the existence of a metric.) The surface integral is over all the boundaries of the integration four-volume, and these include, in addition to the initial t_m and final t_k surfaces, also any walls such as those that might bound the temporal wormholes of Fig. 1. If (as we assume) all such walls are perfectly reflecting, then they give zero contribution to the surface integral, so the only surviving contributions are from t_m and t_k, and the resulting integral conservation law takes the form

$$\sum_k \mathcal{G}^*_{kj} \mathcal{G}_{ki} = \sum_m \mathcal{G}^*_{mj} \mathcal{G}_{mi} \; . \quad (18)$$

Since the times t_m, t_i, and t_j are all in the initial, flat, chronal region, the propagators on the right-hand side of this equation are precisely the flat propagators K^*_{mj} and K_{mi}; and their unitarity brings Eq. (18) into precisely the same expression as Eq. (3). Since the summation (integration) over k in the resulting Eq. (3) is performed at time t_k to the future of the time machine, and the times t_i and t_j are to its past, this equation states that propagation through the time machine is unitary.

However, we must make a cautionary remark about the above theorem. As Tipler and Hawking have shown [3,4], whenever a dischronal region arises in an asymptotically flat spacetime generically, it must be accompanied by some form of spacetime singularity (though in some cases the singularity can be exceedingly mild and irrelevant for physics [7,4]). If in the derivation of Eq. (18) the region we integrate over contains a singularity, then the above derivation, strictly speaking, is invalid. The difficulty can, under a wide range of circumstances, be fixed up by the following method. One simply needs to excise the singular region and construct a boundary around where the singular region was removed. On that boundary, one again sets reflecting boundary conditions, and the theorem we proved above will remain true. What then is the physical significance of such a procedure? In a certain sense it follows naturally from the physical motivation for regarding the path integral as fundamental. We are simply asserting that paths cannot begin or terminate on the singularity. In fact, we have already used this condition implicitly in the analysis of the previous section. The surfaces of constant t that contain V_- and V_+ (Fig. 1) possess "mild" singularities where the wormhole mouths join onto the remainder of the flat space. We dealt with the problem there by simply requiring paths to be continuous and to enter (leave) the wormhole or to (have) avoid(ed) it. These conditions are precisely the reflecting boundary conditions required to make the theorem sketched above work. In fact, any time machine resembling Fig. 1 will necessarily have singularities of at least this type where the wormholes join the chronal region. While it is physically clear how to deal with such a mild type of singularity, it may be that our method fails for more severe types of singularity that one could imagine finding.

In essence, the analysis that we have given in this section shows that unitarity is a consequence of the local conservation law for probability current [Eq. (17)] . This is a nonrelativistic variant of the theorem, by Friedman, Papastamatiou, and Simon [11], that for a free, relativistic scalar field, unitarity of propagation through a time machine follows from conservation of the Klein-Gordon inner product (a theorem that is subject to cautionary remarks about singularities similar to those stated above).

In our nonrelativistic case [14,10], as for a free, relativistic scalar field [11,13], self-interactions will break the conservation law and produce nonunitarity.

IV. CONCLUSIONS

There are two obvious extensions that deserve examination. The first is an extension of our explicit expressions and properties for the propagator, in spacetimes similar to Fig. 1, to the case of relativistic free particles. In fact, it is more or less obvious that the techniques of Sec. II will carry over to the relativistic case with little change. The only significant differences between the relativistic and the nonrelativistic cases are firstly, that particles can only propagate to the future inside the light cone of their starting point (and similarly for antiparticles traveling backward in time) and secondly, there may be numerous particle-antiparticle creation or annihilation events which cause the trajectory of the particle to zigzag. However, neither of these issue will affect the basic structure of the calculations, and it would appear that all one has to do is to replace K_{ji} by the appropriate relativistic propagator in flat spacetime in order to obtain the corresponding relativistic results.

A second, vastly more complex problem has to do with seeking a deeper understanding of the loss of unitarity for interacting systems. A violation of unitarity means that, if one were to attempt computing probabilities in both the initial and final chronal regions using the Hamiltonian formulation's rules (i.e., the Copenhagen interpretation), one would find that probability is not conserved. Hartle [12] and Friedman, Papastamatiou, and Simon [11] respond to this by seeking from the path-integral formalism and other considerations an alternative way of computing probabilities. From their alternative way (which the fourth author of this paper finds compelling), they conclude that, although one recovers the usual Hamiltonian formulation of quantum mechanics from the path-integral formulation to the future of the time machine, one cannot do so to the past, even though the past region is chronal.

To summarize, we have seen in this paper that, at least at the level of our analysis, there is no contradiction between the postulates of quantum mechanics and the possible existence of causality violation in general relativity. This is quite distinct from the situation in classical mechanics, where the inability to solve uniquely an apparently sensible initial value problem leads to a breakdown of classical predictability. We believe that it is quite possible that a suitable quantum mechanical treatment will resolve many of the classical difficulties of causality violation under wider assumptions than those treated here.

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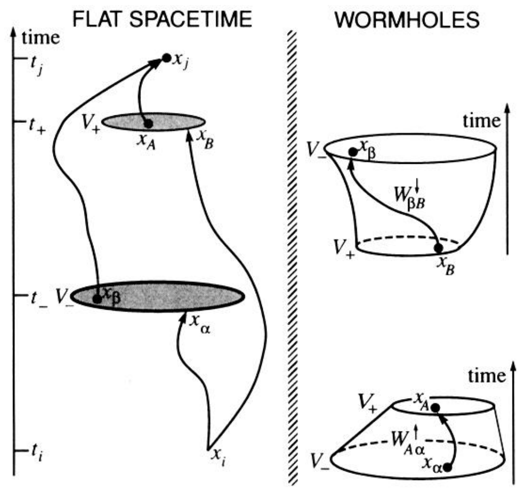

FIG. 1. A nonrelativistic time-machine spacetime and two paths in the spacetime that contribute to the propagator $\mathcal{G}_{ji}$ for a noninteracting, nonrelativistic particle. Left: The external, nonrelativistic, flat part of the spacetime. Right: The temporal wormholes which connect the regions V_- and V_+.